

# CO-FROBENIUS COALGEBRAS

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**ABSTRACT.** We investigate left and right co-Frobenius coalgebras and give equivalent characterizations which prove statements dual to the characterizations of Frobenius algebras. We prove that a coalgebra is left and right co-Frobenius if and only if  $C \cong \text{Rat}(C_{C^*}^*)$  as right  $C^*$ -modules and also that this is equivalent to the fact that the functors  $\text{Hom}_K(-, K)$  and  $\text{Hom}_{C^*}(-, C^*)$  from  $\mathcal{M}^C$  to  ${}_{C^*}\mathcal{M}$  are isomorphic. This allows a definition of a left-right symmetric concept of co-Frobenius coalgebras that is perfectly dual to the one of Frobenius algebras and coincides to the existing notion left and right co-Frobenius coalgebra.

## INTRODUCTION

Let  $A$  be an algebra.  $A$  is called Frobenius algebra if the right regular module  $A$  is isomorphic to the right  $A$  module  $A^*$ . This is known to be equivalent to the fact that the functors  $\text{Hom}_A(-, A)$  and  $\text{Hom}_K(-, K)$  from  $\mathcal{M}_A$  to  ${}_A\mathcal{M}$  are naturally isomorphic (see [CR]). A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is called Frobenius if it has the same left and right adjoint (see [CMZ], [NT1]). By using this concept, a Frobenius algebra can equivalently be characterized by the fact that the forgetful functor  $F$  from  $\mathcal{M}_A$  to  $\mathcal{M}_K$  is a Frobenius functor. Moreover, an algebra  $A$  is Frobenius if and only if there is a  $K$  bilinear form  $(-, -)$  on  $A \times A$  that is associative and (left) non-degenerate. These definitions turn out to be left-right symmetric and imply the finite dimensionality of the algebra  $A$ .

Let  $C$  be a coalgebra over a field  $K$  and let  $C^*$  be the dual coalgebra. The coalgebra  $C$  is called *right co-Frobenius* if there is a monomorphism of right  $C^*$  modules from  $C$  into  $C^*$ , or equivalently, there is a  $C^*$  balanced bilinear form on  $C \times C$  which is right non-degenerate. The concept of *left co-Frobenius coalgebra* is defined by symmetry. A coalgebra is called simply co-Frobenius if it is both left and right co-Frobenius. The notion of right co-Frobenius coalgebra was introduced by Lin in [L] and it abstracts a relevant coalgebra property of Hopf algebras with non-zero integral ([L], Theorem 3). A study of one side co-Frobenius coalgebras and more generally of one side quasi co-Frobenius coalgebras was carried out in [NT1] and [NT2]. For a Hopf algebra  $H$ , it is known that  $H$  is right co-Frobenius is equivalent to  $H$  having non-zero right integral, so this property defines this class of Hopf algebras. Hopf algebras with non-zero integrals are very important and have been intensely studied in the last years, mainly because they have very good structural

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and representation theoretic properties (see [AD], [DNR], [H1], [S1], [S2], [Sw1], [Sw2] and references therein). Quantum groups with non-zero integrals are also of great interest; see [APW], [AD], [H1], [H2], [H3].

Lin proves that for a Hopf algebra  $H$  being right co-Frobenius amounts to being left co-Frobenius and therefore it defines a left-right symmetric concept for Hopf algebras. However, that is not the case for coalgebras, as an example in [L] shows that right co-Frobenius coalgebras need not be left co-Frobenius (see also [DNR], example 3.3.7). Therefore the question of whether a left-right symmetric concept of co-Frobenius coalgebra can be defined, a concept that would be dual to that of Frobenius algebras, would recover the notion of Frobenius algebra in the finite dimensional case and the notion co-Frobenius coalgebra. It is proved in [NT2] that a coalgebra  $C$  is co-Frobenius if and only if  $\text{Rat}(C^*C^*) \cong {}_C C^*$  and  $\text{Rat}(C^*_{C^*}) \cong C^*_{C^*}$ . It is then natural to ask which coalgebras satisfy the property  $\text{Rat}(C^*C^*) \cong {}_C C^*$  and if this property  $\text{Rat}(C^*C^*) \cong {}_C C^*$  and if these are exactly the co-Frobenius coalgebras. In order to be able to make an analogue to the algebra case, we will consider the duality functors  $C^*$ -dual  $\text{Hom}_{C^*}(-, C^*)$  and  $K$ -dual  $\text{Hom}(-, K)$  from  $\mathcal{M}^C$  to  ${}_{C^*}\mathcal{M}$ . For the dual of the right regular comodule  $C$  it is natural to look at  $C^*$  and take its rational part  $\text{Rat}(C^*_{C^*})$ .

The main result of the paper states that if  $C^C$  is isomorphic to the dual  $\text{Rat}(C^*C^*)$  of the left comodule  ${}^C C$ , then the coalgebra  $C$  is co-Frobenius and thus this property defines the notion of co-Frobenius coalgebra. As a consequence, we also show that  $C$  is co-Frobenius if and only if the two duality functors  $\text{Hom}_{C^*}(-, C^*)$  and  $\text{Hom}(-, K)$  are isomorphic when evaluated on right comodules and furthermore, this is also equivalent to the existence of a bilinear form on  $C$  which is  $C^*$ -balanced and is left and right non-degenerate. These results generalize and extend known results from the algebra case; however, the proofs are completely different from the ones in the algebra case and involve the use of several techniques and results specific to coalgebra theory.

## 1. PRELIMINARY RESULTS

We recall some results on coalgebras, which we state with references for completeness of the text. For basic facts on coalgebra and comodule theory one should see [A], [DNR], [M] and [Sw1]. Let  $C$  be a coalgebra. If  $M$  is a finite dimensional right  $C$ -comodule, then it becomes left  $C^*$ -module by  $c^* \cdot m = m_0 c^*(m_1)$  and its dual  $M^* = \text{Hom}(M, K)$  becomes a right  $C^*$ -module (as stated above) which is rational, so it is a left  $C$ -comodule. The following results are Proposition 4, page 34 from [D] and Lemma 15 from [L] respectively. See also Corollaries 2.4.19 and 2.4.20 from [DNR].

**Lemma 1.1.** *Let  $Q$  be a finite dimensional right  $C$ -comodule. Then  $Q$  is injective (projective) as a left  $C^*$ -module if and only if it is injective (projective) as a right  $C$ -comodule.*

**Lemma 1.2.** *Let  $M$  be a finite dimensional right  $C$ -comodule. Then  $M$  is an injective right  $C$ -comodule if and only if  $M^*$  is a projective left  $C$ -comodule.*

Let  $C$  be a coalgebra over a field  $K$ . Denote by  $\mathcal{L}$  and respectively  $\mathcal{R}$  a set of representatives for the types of isomorphism of simple left, respectively right  $C$ -comodules. Let  $s({}^C C) = \bigoplus_{i \in I} S_i$  be a decomposition of the left socle of  $C$  into simple left comodules and  $s(C^C) = \bigoplus_{j \in J} T_j$  a decomposition of the right socle. Then we have a direct sum decomposition  $C = \bigoplus_{i \in I} E(S_i)$  as left  $C$ -comodules and  $C = \bigoplus_{j \in J} E(T_j)$  as right  $C$ -comodules, where  $E(S_i)$  are injective envelopes of  $S_i$  contained in  $C$  and  $E(T_j)$  are right comodules that are injective envelopes of  $T_j$  contained in  $C$ . Also every simple left  $C$ -comodule is isomorphic to one of the  $S_i$ 's and similarly every right simple  $C$ -comodule is isomorphic to one of the  $T_j$ 's. Note that the sets  $\mathcal{L}$ ,  $\mathcal{R}$ ,  $I$  and  $J$  are of the same cardinality. See [G] and [DNR].

Let  $C$  be a coalgebra. Recall from [L] that  $C$  is said to be right co-Frobenius if there is a monomorphism of right  $C^*$ -modules from  $C$  into  $C^*$ . The notion of left co-Frobenius algebra is defined similarly.  $C$  is called right (left) semiperfect if every finite dimensional right (left) comodule has a projective cover. Also recall from [L] that  $C$  is right semiperfect if and only if the injective envelope of any simple left comodule is finite dimensional, and that a right co-Frobenius coalgebra is also right semiperfect.

The following proposition shows that  $C \cong \text{Rat}_C(C^*)$  as left  $C^*$ -modules for a left and right co-Frobenius coalgebra. See Corollary 1.2 from [CDN] for the proof.

**Proposition 1.3.** *If  $C$  is a left and right co-Frobenius coalgebra, then any injective morphism of right  $C$  modules  $\varphi : C \rightarrow (\text{Rat}_C(C^*))$  is an isomorphism.*

**Lemma 1.4.** *Let  $S$  be a simple left comodule and  $E(S)$  be an injective envelope of  $S$  contained in  $C$ . Then  $S^\perp = \{\alpha \in E(S)^* \mid \alpha|_S = 0\}$  is a maximal and small left  $C^*$  submodule of  $E(S)^*$  and  $E(S)^*$  is generated by any  $f \notin S^\perp$ . Consequently,  $E(S)^*$  is an indecomposable left  $C^*$ -module.*

**Proof.** We begin by the following remark: for any left comodule  $M$ , we have a left  $C^*$ -modules isomorphism  $\text{Hom}^C(M, C) \cong \text{Hom}(M, K)$  given by  $f \mapsto \varepsilon \circ f$ , where  $\varepsilon$  is the counit of  $C$ . Here the left  $C^*$ -module structure on  $\text{Hom}^C(M, C)$  is given by the left  $C^*$ -action on  $C$ , namely  $(c^* \cdot f)(x) = c^* \cdot f(x) = c^*(f(x)_2)f(x)_1$ . For  $M = C$  this isomorphism becomes even an isomorphism of algebras  $(\text{Hom}^C(C, C), +, \circ) \cong (C^*, +, *)$ , where  $\circ$  is the composition of morphisms and  $*$  is the usual convolution on  $C^*$ . It is not difficult to see that by this isomorphism  $\text{Hom}^C(M, C)$  becomes a left  $\text{Hom}^C(C, C)$  module and that the structure is given simply by composition, namely for  $\alpha \in \text{Hom}^C(C, C)$  and  $f \in \text{Hom}^C(M, C)$ ,  $(\alpha \rightarrow f) = \alpha \circ f$ . Thus we may proof the statement equivalently for the left  $\text{Hom}^C(C, C)$  module  $\text{Hom}^C(M, C)$ .

Let  $S$  be a simple subcomodule of  $C$  and  $E(S)$  an injective envelope of  $S$  contained in  $C$ . Then there is a left subcomodule  $X$  of  $C$  such that  $E(S) \oplus X = C$  in  ${}^C\mathcal{M}$ . As the functor  $\text{Hom}^C(-, C)$  is exact, we obtain an epimorphism  $\pi : \text{Hom}(E(S), C) \rightarrow \text{Hom}(S, C)$ ,  $\pi(f) = f|_S$ . The kernel of this morphism coincides with  $S^\perp$ . Let  $f \in \text{Hom}^C(E(S), C)$  such that  $f|_S \neq 0$ . Then  $\text{Ker}(f) \cap S = 0$  as  $S$  is simple and so  $\text{Ker } f = 0$ .

because  $S$  is essential in  $E(S)$ . So  $E(S) \cong f(E(S))$  and thus there is a left subcomodule  $M$  of  $C$  so that  $C = f(E(C)) \oplus M$ . We can extend  $f$  to a left comodule isomorphism  $\bar{f}$  from  $C$  to  $C$  by taking  $\bar{f}$  to be  $f$  on  $E(S)$  and denote by  $h$  its inverse. Now if  $g$  is another element of  $\text{Hom}^C(E(S), C)$ , denoting by  $\bar{g}$  its extension to  $C$  that equals 0 on  $X$  and  $g$  on  $E(S)$  we have  $\bar{g} = \bar{g} \circ \text{id}_C = \bar{g} \circ (h \circ \bar{f}) = (\bar{g} \circ h) \circ \bar{f}$  and then restricting to  $E(S)$ ,  $g = (\bar{g} \circ h) \circ f$ , equivalently  $g = (\bar{g} \circ h) \rightarrow f$  (by the left action of  $\text{End}^C(C)$  on  $\text{Hom}^C(E(S), C)$ ) showing that  $\text{Hom}^C(E(S), C)$  is generated by  $f$ , for an arbitrary  $f \in \text{Hom}^C(E(S), C) \setminus S^\perp$ . This shows that if  $M \subsetneq E(S)^*$  is a submodule of  $E(S)^*$ , then  $M \subseteq S^\perp$ , so  $S^\perp$  is the only maximal subcomodule of  $E(S)^*$ , and also  $S^\perp \ll E(S)^*$ . Consequently if  $E(S)^* = M \oplus N$ , then if  $M, N \neq E(S)^*$  we get  $M, N \subset S^\perp$  which is a contradiction as  $S^\perp \neq E(S)^*$  as  $S \neq 0$ .  $\square$

## 2. THE MAIN RESULT

**Proposition 2.1.** *Suppose  $C \cong \text{Rat}(C_{C^*}^*)$ . Then for every  $T \in \mathcal{R}$ , we have either  $E(T)$  finite dimensional and  $E(T)^* \cong E(S)$  for some  $S \in \mathcal{L}$ , or  $E(T)$  is infinite dimensional and  $\text{Rat}(E(T)^*) = 0$ .*

**Proof.** From the hypothesis,  $C$  is right co-Frobenius and hence right semiperfect and then the  $E(S_i)$ 's are finite dimensional. Then we have

$$\bigoplus_{i \in I} E(S_i) \cong \text{Rat}(C_{C^*}^*) = \text{Rat}\left(\prod_{j \in J} E(T_j)^*\right) = \text{Rat}(E(T_k)^*) \oplus \text{Rat}\left(\prod_{j \in J \setminus \{k\}} E(T_j)^*\right)$$

for any  $k \in J$ . Then  $\text{Rat}(E(T_k)^*)$  is injective and isomorphic to a direct sum of  $E(S_i)$ . By Lemma 1.1 we have that the  $E(S_i)$ 's are injective in  $\mathcal{M}_{C^*}$  and by Lemma 1.4  $E(T_k)^*$  is indecomposable. Then we have two possibilities:

-  $\text{Rat}(E(T_k)^*) = 0$  which implies that  $E(T)$  is infinite dimensional (as otherwise  $E(T)^*$  is finite dimensional and rational)

-  $\text{Rat}(E(T_k)^*) \neq 0$  and then there is a direct sum decomposition  $\bigoplus_{i \in I'} E(S_i) \cong \text{Rat}(E(T_k)^*)$  so there is an  $i \in I$  such that  $E(S_i)$  is a direct summand of  $E(T_k)^*$  (because  $E(S_i)$  is injective in  $\mathcal{M}_{C^*}$ ) and then  $E(S_i) \cong E(T_k)^*$  (because  $E(T_k)^*$  is an indecomposable right  $C^*$ -module). As every  $T \in \mathcal{R}$  is isomorphic to one of the  $T_j$ 's, the proposition is proved.  $\square$

Denote by  $J_0 = \{j \in J \mid \text{Rat}(E(T_j)^*) \neq 0\}$  and  $J' = J \setminus J_0$ . Notice that  $\text{Rat}\left(\prod_{j \in J'} E(T_j)^*\right) =$

0. Indeed, denoting by  $p_j$  the canonical projection on the  $j$ 'th component of the direct product, we have that if  $0 \neq x \in \text{Rat}\left(\prod_{j \in J'} E(T_j)^*\right)$ , then there is  $j \in J$  such that  $p_j(x) \neq 0$ . But  $p_j(x) \in \text{Rat}(E(T_j)^*) = 0$ , which is a contradiction.

**Corollary 2.2.** *With the above notations we have  ${}^C C \cong \text{Rat}\left(\prod_{j \in J_0} E(T_j)^*\right)$  provided that  $C \cong \text{Rat}(C_{C^*}^*)$ .*

**Proposition 2.3.** *If  $C$  is right semiperfect then the set  $\{E(S)^* \mid S \in \mathcal{L}\}$  is a family of generators of  $\mathcal{M}^C$ .*

**Proof.** For any  $S \in \mathcal{L}$  we have that  $E(S)$  is finite dimensional and then  $E(S)^* \in \mathcal{M}^C$ . It is enough to prove that the  $E(S)^*$  generate the finite right comodules. If  $M$  is such a comodule, then  $M^*$  is a finite dimensional left comodule, thus there is a monomorphism  $0 \rightarrow M^* \xrightarrow{u} \bigoplus_{\alpha \in F} E(S_\alpha)$  with  $F$  a finite set and  $S_\alpha$  simple left comodules. Taking duals, we obtain an epimorphism  $\bigoplus_{\alpha \in F} E(S_\alpha)^* \xrightarrow{u^*} M^{**} \cong M \rightarrow 0$  in  $\mathcal{M}^C$ .  $\square$

**Proposition 2.4.** *Let  $E(T)$  be an infinite dimensional injective indecomposable right comodule. Suppose that there is an epimorphism  $E \xrightarrow{\pi} E(T) \rightarrow 0$ , such that  $E = \bigoplus_{\alpha \in A} E_\lambda$  and  $E_\lambda$  are finite dimensional injective right comodules. Then there is an epimorphism from a direct sum of finite dimensional injective right comodules to  $E(T)$  with kernel containing no non-zero injective comodules.*

**Proof.** Denote  $H = \text{Ker } \pi$  and consider the set  $\mathcal{N} = \{Q \subset H \mid Q \text{ is an injective comodule}\}$ . We see that  $\mathcal{N} \neq \emptyset$  as  $0 \in \mathcal{N}$  and that  $\mathcal{N}$  is an inductive ordered set. To see this consider a chain  $(X_i)_{i \in L}$  of elements of  $\mathcal{N}$  and  $X = \bigcup_{i \in L} X_i$  which is a subcomodule of  $H$ . Let  $s(X) = \bigoplus_{\lambda \in \Lambda} S_\lambda$  be a decomposition into simple subcomodules of the socle of  $X$ . Then  $s(X)$  is essential in  $X$  and for every  $\lambda \in \Lambda$  there is an  $i = i(\lambda) \in L$  such  $S_\lambda \subset X_{i(\lambda)}$ . As  $X_i$  is injective, there is an injective envelope  $H_\lambda$  of  $S_\lambda$  that is contained in  $X_i$ . First we prove that the sum  $\sum_{\lambda \in \Lambda} H_\lambda$  is direct. To see this it is enough to prove that  $H_{\lambda_0} \cap (\sum_{\lambda \in F} H_\lambda) = 0$ , for every finite subset  $F \subseteq \Lambda$  and  $\lambda_0 \in \Lambda \setminus F$ . We prove this by induction on the cardinal of  $F$ . If  $F = \{\lambda\}$  then  $H_{\lambda_0} \cap H_\lambda = 0$  because otherwise we would have  $S_\lambda = S_{\lambda_0}$ , a contradiction. If the statement is proved for all sets with at most  $n$  elements and  $F$  is a set with  $n+1$  elements then the sum  $\sum_{\lambda \in F} H_\lambda$  is direct, because  $F = (F \setminus \{\lambda'\}) \cup \{\lambda'\}$  for every  $\lambda' \in F$  and we apply the induction hypothesis. If  $H_{\lambda_0} \cap (\sum_{\lambda \in F} H_\lambda) \neq 0$  we get that  $S_{\lambda_0} \subseteq \sum_{\lambda \in F} H_\lambda$ , because  $S_{\lambda_0}$  is essential in  $H_{\lambda_0}$ . But as the sum  $\sum_{\lambda \in F} H_\lambda$  is direct we have that  $s(\sum_{\lambda \in F} H_\lambda) = s(\bigoplus_{\lambda \in F} H_\lambda) = \bigoplus_{\lambda \in F} s(H_\lambda) = \bigoplus_{\lambda \in F} S_\lambda$  so  $S_{\lambda_0} \subset s(\sum_{\lambda \in F} H_\lambda) = \bigoplus_{\lambda \in F} S_\lambda$  which is a contradiction with  $\lambda_0 \notin F$ . Now notice that  $X = \bigoplus_{\lambda \in \Lambda} H_\lambda$ . Since  $\bigoplus_{\lambda \in \Lambda} H_\lambda$  is injective, it is a direct summand of  $X$ . Write  $X = (\bigoplus_{\lambda \in \Lambda} H_\lambda) \oplus H'$  and suppose  $H' \neq 0$ . Take  $S' \subseteq H'$  a simple subcomodule of  $H'$ . Then  $S' \subseteq s(X) = \bigoplus_{\lambda \in \Lambda} S_\lambda \subseteq \bigoplus_{\lambda \in \Lambda} H_\lambda$  which is a contradiction. We conclude that  $X$  is injective, thus  $X \in \mathcal{N}$ .

By Zorn's Lemma we can then take  $M$  a maximal element of  $\mathcal{N}$ . As  $M$  is an injective comodule, it is a direct summand of  $H$  and take  $M \oplus H' = H$ . It is obvious that  $H$  is essential in  $E = \bigoplus_{\alpha \in A} E_\alpha$ , because otherwise taking  $E(H)$  an injective envelope of  $H$  contained in

$E$ , we would have  $E(H) \oplus Q = E$  so  $E(T) \cong \frac{E(H) \oplus Q}{H} \cong \frac{E(H)}{H} \oplus Q$  which is a contradiction as  $Q$  is a direct sum of finite dimensional comodules and  $E(T)$  is indecomposable infinite dimensional. Take  $E'$  an injective envelope of  $H'$  contained in  $E$ . If  $M \oplus E' \subsetneq E$  then there is a simple comodule  $S$  contained in  $E$  and such that  $S \cap (M \oplus E') = 0$ , because  $M \oplus E'$  is a direct summand of  $E$  as it is injective. Then  $S \cap H = 0$  ( $H \subseteq M \oplus E'$ ), which contradicts the fact that  $H \subseteq E$  is an essential extension. Consequently,  $M \oplus E' = E$  and then

$$E(T) \cong \frac{E}{H} = \frac{M \oplus E'}{M \oplus H'} = \frac{E'}{H'}$$

where  $E'$  is a direct sum of finite dimensional injective indecomposable modules and  $H'$  does not contain non-zero injective modules because of the maximality of  $M$ .  $\square$

Recall from [T] that a left  $C$ -comodule  $M$  is called quasi-finite if and only if  $\text{Hom}^C(S, M)$  is finite dimensional for every  $S \in \mathcal{L}$ , equivalently, if  $s(M) = \bigoplus_{l \in L} M_l$  is a decomposition of  $M$  into simple left comodules then the set  $\{l \in L : M_l \cong S\}$  is finite for every  $S \in \mathcal{L}$ .

**Lemma 2.5.** *Let  $(X_i)_{i \in L}$  be a family of nonzero (right)  $C$ -comodules such that  $\Sigma = \bigoplus_{i \in L} X_i$*

*is a quasifinite module. Then  $\bigoplus_{i \in L} X_i = \prod_{i \in L}^C X_i$ , where  $\prod_{i \in L}^C$  is the product in the category of comodules.*

**Proof.** We have that  $\prod_{i \in L}^C X_i = \text{Rat}(\prod_{i \in L} X_i)$ , where  $\prod_{i \in L}$  is the product of modules. Suppose that  $x = (x_i)_{i \in L} \in P = \text{Rat}(\prod_{i \in L} X_i)$  and the set  $L' = \{i \in L \mid x_i \neq 0\}$  is infinite. Then  $C^* \cdot x$  is a finite dimensional rational module, so it has a finite composition series. For each  $i \in L$  the canonical projection  $p_i : C^* \cdot x \rightarrow C^* \cdot x_i$  is an epimorphism, thus  $C^* \cdot x_i$  is a rational module. For every  $i \in L'$  consider  $S_i$  a simple subcomodule of  $C^* \cdot x_i$ . As  $\Sigma$  is quasi-finite and  $L'$  is infinite, we have that the set  $\mathcal{R}' = \{T \in \mathcal{R} \mid \exists i \in L' \text{ such that } S_i \cong T\}$  is infinite ( $\mathcal{R}$  is the chosen set of representatives for the types of isomorphisms of simple right  $C$ -comodules). For each  $T \in \mathcal{R}'$  choose  $k \in L'$  such that  $T \cong S_k$ . Denote by  $\Lambda$  the set of these  $k$ 's. As for every  $T \in \mathcal{R}'$  there is  $k$ , a monomorphism  $T \hookrightarrow C^* \cdot x_k$  and an epimorphism  $C^* \cdot x \xrightarrow{p_k} C^* \cdot x_k$ , it follows then that every composition series of  $C^* \cdot x$  contains a simple factor isomorphic to  $T$ . As  $C^* \cdot x$  is finite dimensional it follows that the set  $\mathcal{F}$  of simple left  $C^*$ -modules appearing as factors in any composition series is finite. But  $\mathcal{R}' \subseteq \mathcal{F}$  which is a contradiction to the fact that  $\mathcal{R}'$  is infinite. Thus  $x \in \bigoplus_{i \in L} X_i$ , and then  $\text{Rat}(\prod_{i \in L} X_i) \subseteq \bigoplus_{i \in L} X_i$  and the proof is finished as the converse inclusion is obviously true.  $\square$

**Theorem 2.6.** *Let  $C$  be a coalgebra such that  $C \cong \text{Rat}(C_{C^*}^*)$  as right  $C^*$ -modules (left  $C$ -comodules). Then  $C$  is left semiperfect and  $C \cong \text{Rat}(C_* C^*)$  as left  $C^*$ -modules.*

**Proof.** Let  $T$  be a simple right  $C$ -comodule such that  $E(T)$  is infinite dimensional. Then  $\text{Rat}(E(T)^*) = 0$  by proposition 2.1. We have that  $C$  is right co-Frobenius, thus it is also



right semiperfect. By Proposition 2.3 there is an exact sequence of right comodules

$$0 \rightarrow H \hookrightarrow \bigoplus_{\alpha \in A} E(S_\alpha)^* \xrightarrow{u} E(T) \rightarrow 0$$

with  $E(S_\alpha)$  finite dimensional injective left comodules. Let  $E = \bigoplus_{\alpha \in A} E(S_\alpha)^*$ . As  $E(S_\alpha)$  are finite dimensional injective left comodules, they are injective also as right  $C^*$ -modules (by Lemma 1.1) and as  $C \cong \text{Rat}(C_{C^*}^*)$  it follows that every injective indecomposable comodule is a direct summand of  $C^*$ , thus it is projective. By Lemma 1.2 we obtain that every  $E_\alpha = E(S_\alpha)^*$  is also injective, and also finite dimensional indecomposable and then by Proposition 2.4 we may assume that  $H$  does not contain nonzero injective comodules. Take  $n \in A$  and set  $E' = \bigoplus_{\alpha \in A \setminus \{n\}} E_\alpha$ . Then  $H + E' = E$ . To see this first notice that  $H + E'$  has finite codimension in  $E$ . There is an epimorphism of right comodules (thus of left  $C^*$ -modules)

$$E(T) \cong \frac{E}{H} \rightarrow \frac{E}{H + E'} \rightarrow 0$$

and by taking duals we get a monomorphism of right  $C^*$ -modules

$$0 \rightarrow \left( \frac{E}{H + E'} \right)^* \rightarrow \left( \frac{E}{H} \right)^* \cong E(T)^*$$

But the dual of the finite dimensional right comodule  $E/(H + E')$  is a rational right  $C^*$ -module, implying that  $\text{Rat}(E(T)^*) \neq 0$  if  $H + E' \neq E$ , a contradiction.

By the isomorphisms

$$E_n \cong \frac{E}{E'} = \frac{H + E'}{E'} \cong \frac{H}{H \cap E'}$$

we conclude that there is an epimorphism from  $H$  onto  $E_n$ . This morphism must split, as  $E_n$  is also a projective right comodule (again by Lemma 1.2). This shows that  $H$  contains an injective subcomodule isomorphic to  $E_n$ , which contradicts the supposition that  $H$  does not contain nonzero injective subcomodules. Hence  $C$  is semiperfect.

Now we have  $\bigoplus_{i \in I} E(S_i) \cong \text{Rat}(C_{C^*}^*) \cong \text{Rat}(\prod_{j \in J_0} E(T_j)^*)$ . But  $E(T_j)^*$  are indecomposable and also injective finite dimensional left comodules; write  $L_j = E(T_j)^*$ . Then  $L_j$  is the injective envelope of its socle, so  $L_j \cong L_{j'}$  if and only if  $s(L_j) \cong s(L_{j'})$  (and equivalently,  $E(T_j) = L_j^* \cong L_{j'}^* = E(T_{j'})$ ). This shows that for any  $S \in \mathcal{L}$ , there are only finitely many  $j \in J$  with the property  $s(L_j) \cong S$ , because only finitely many  $E(T_j)$ 's can be isomorphic to the same injective indecomposable. This shows that the comodule  $\bigoplus_{j \in J} L_j$  is quasifinite, and then by Lemma 2.5 we have that  $\bigoplus_{i \in I} E(S_i) \cong \text{Rat}(\prod_{j \in J} E(T_j)^*) \cong \bigoplus_{j \in J} E(T_j)^*$ . By Krull-Remak-Schmidt-Azumaya's Theorem, there is a bijection  $\varphi : I \rightarrow J$  such that  $E(S_i) \cong E(T_{\varphi(i)})^*$  for every  $i \in I$ , equivalently,  $E(S_i)^* \cong E(T_{\varphi(i)})$  for all  $i$ . We then obtain that the comodule  $\bigoplus_{j \in J} E(T_j) \cong \bigoplus_{i \in I} E(S_i)^*$  is quasifinite, and again by Lemma 2.5 we have that  $\bigoplus_{j \in J} E(T_j) \cong \text{Rat}(\prod_{i \in I} E(S_i)^*)$ , finally showing that  $C \cong \text{Rat}(C^* C^*)$  as left  $C^*$ -modules.  $\square$

**Remark 2.7.** Let  $\mathcal{C}$  be a category. Then for every two objects  $X, Y$ , we consider the "Yoneda" bijection of sets  $\Lambda : \underline{\mathbf{Nat}}(\mathrm{Hom}(-, X), \mathrm{Hom}(-, Y)) \longrightarrow \mathrm{Hom}(X, Y)$  defined by  $\Lambda(\varphi) = \varphi_X(1_X)$  with inverse  $\Lambda^{-1}(\theta) = (f \mapsto \theta \circ f)$ . Moreover, if  $\varphi$  is a natural equivalence with inverse  $\varphi'$ , then  $\theta = \varphi_X(1_X) : X \rightarrow Y$  is an isomorphism with inverse  $\theta' = \varphi'_Y(1_Y) : Y \rightarrow X$ .

**Theorem 2.8.** Let  $C$  be a coalgebra. Then the following assertions are equivalent:

- (i)  $C$  is a co-Frobenius coalgebra (left and right co-Frobenius).
- (ii)  $\mathrm{Rat}(C^*C^*) \cong C$  in  $\mathcal{M}^C$  (or as left  $C^*$ -modules).
- (iii) The functors  $\mathrm{Hom}_K(-, K)$  and  $\mathrm{Hom}_{C^*}(-, C^*)$  from  $\mathcal{M}^C \subset {}_{C^*}\mathcal{M}$  to  $\mathcal{M}_{C^*}$  are naturally equivalent.
- (iv) The left hand side versions of (ii) and (iii).
- (v) There is a  $K$ -bilinear form  $(-, -)$  on  $C \times C$  that is  $C^*$ -balanced (i.e.  $(c \cdot h^*, d) = (c, h^* \cdot d)$  for all  $c, d \in C$  and  $h^* \in C^*$ ) and left and right non-degenerate.

**Proof.** (ii) $\Leftrightarrow$ (iii) We have a natural equivalence  $\mathrm{Hom}^C(-, C) \cong \mathrm{Hom}(-, K) : \mathcal{M}^C \rightarrow \mathcal{M}_{C^*}$ ,  $h \mapsto \varepsilon \circ h$ , where  $\varepsilon$  is the counit of  $C$  and also a natural equivalence of functors  $\mathrm{Hom}_{C^*\mathcal{M}}(-, C^*) \cong \mathrm{Hom}^C(-, C^*) \cong \mathrm{Hom}^C(-, \mathrm{Rat}(C^*C^*))$ . Thus by the previous remark the functors  $\mathrm{Hom}(-, K)$  and  $\mathrm{Hom}_{C^*}(-, C^*)$  from  $\mathcal{M}^C$  to  $\mathcal{M}_{C^*}$  are isomorphic if and only if there is an isomorphism of left  $C^*$ -modules (right  $C$ -comodules)  $C \cong \mathrm{Rat}(C^*C^*)$ .

(i) $\Rightarrow$ (ii) follows from Proposition 1.3 and (ii) $\Rightarrow$ (i) from Theorem 2.6.

(v) $\Leftrightarrow$ (i) For an isomorphism  $\varphi : C \rightarrow \mathrm{Rat}(C^*C^*)$  one can define  $(-, -) : C \times C \rightarrow C$  by  $(c, d) = \varphi(d)(c)$ . This is a  $C^*$ -balanced form by the morphism property of  $\varphi$ , is left non-degenerate by the injectivity of  $\varphi$  and right nondegenerate by the density of  $\mathrm{Im}\varphi = \mathrm{Rat}(C^*C^*)$  in  $C^*$  in the finite topology on  $C^*$ , because  $\mathrm{Rat}(C^*C^*) = \bigoplus_{i \in I} E(S_i)^*$  (as shown above in the proof of Theorem 2.6) which is dense in  $\prod_{i \in I} E(S_i)^*$ . For the converse consider  $\varphi : C \rightarrow C^*$  and  $\psi : C \rightarrow C^*$  defined by  $\varphi(c)(d) = (c, d)$  and  $\psi(c)(d) = (d, c)$  for all  $c, d \in C$ . Then an easy computation shows that  $\varphi$  is an injective morphism of right  $C^*$ -modules and  $\psi$  is an injective morphism of left comodules, thus  $C$  is left and right co-Frobenius.  $\square$

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